2. Reductive Perturbation Method

Small-amplitude oscillations near the Hopf bifurcation point are generally governed by a simple evolution equation. If such oscillators form a field through diffusion-coupling, the governing equation is a simple partial differential equation called the Ginzburg-Landau equation.

2.1 Oscillators Versus Fields of Oscillators

Many theories on the nonlinear dynamics of dissipative systems are based on the first-order ordinary differential equations

$$\frac{dX_i}{dt} = F_i(X_1, X_2, \ldots, X_n; \mu), \qquad i = 1, 2, \ldots, n,$$

which include some parameters represented by μ; a more convenient vector form

$$\frac{dX}{dt} = F(X; \mu) \tag{2.1.1}$$

is sometimes preferred. As a specific example, we mention the dynamics of chemical reaction systems which are maintained uniformly in space. In this case, X usually represents a set of concentrations of the chemical species involved, and μ may be taken to be the flow rate at which certain chemicals are constantly fed into the system so that their consumption due to reactions may be compensated.

For some range of μ, the system may stay stable in a time-independent state. In particular, this is usually the case for macroscopic physical systems which lie sufficiently close to thermal equilibrium. In many systems, such a steady state loses stability at some critical value μ_c of μ, and beyond it (say $\mu > \mu_c$), gives way to periodic motion. In the parameter-amplitude plane, this appears as a branching of time-periodic solutions from a stationary solution branch, and this phenomenon is generally called the *Hopf bifurcation*. For various mathematical aspects of the Hopf bifurcation, one may refer to the book by Marsden and McCracken (1976). In chemical reactions, the corresponding phenomenon is called the onset of *chemical oscillations*. Besides chemical reactions, one may point out many examples from electrical and mechanical engineering, optics, biology, biochemistry, and possibly some other fields, for which ordinary-dif-

ferential-equation models form a natural basis for mathematical analysis, so that the appearance of oscillations may be understood in the way stated above.

As μ increases further, the system may show more and more complicated dynamics through a number of bifurcations. It may show complicated periodic oscillations, quasi-periodic oscillations or a variety of non-periodic behaviors. For instance, we know of the recent discoveries of fantastic bifurcation structures in the spatially homogeneous Belousov-Zhabotinsky reaction, see Hudson et al., 1979.

Coming back to limit cycle oscillations shown by systems of ordinary differential equations, this simple mode of motion still seems to deserve some more attention, especially in relation to its role as a basic functional unit from which various dynamical complexities arise. This seems to occur in at least two ways. As mentioned above, one may start with a simple oscillator, increase μ, and obtain complicated behaviors; this forms, in fact, a modern topic. However, another implication of this dynamical unit should not be left unnoticed. We should know that a limit cycle oscillator is also an important component system in various self-organization phenomena and also in other forms of spatio-temporal complexity such as turbulence. In this book, particular emphasis will be placed on this second aspect of oscillator systems. This naturally leads to the notion of the "many-body theory of limit cycle oscillators"; we let many oscillators contact each other to form a "field", and ask what modes of self-organization are possible or under what conditions spatio-temporal chaos arises, etc. A representative class of such many-oscillator systems in theory and practical application is that of the fields of diffusion-coupled oscillators (possibly with suitable modifications), so that this type of system will primarily be considered in this book.

In any case, we should begin with some investigation of the component systems, i.e., limit cycle oscillators. Although the specific feature of limit cycle oscillations (e.g., orbital forms, oscillation patterns, etc.) may vary greatly from system to system, there exists one remarkable universal fact, namely, that all systems come to behave in a similar manner sufficiently close to the onset of oscillations. Mathematicians may say that this is a consequence of the *center manifold theorem*. More physically, we are left with only a couple of relevant dynamical variables close to criticality, whose time scales are distinguishably slower than those of the remaining variables, so that the latter can be eliminated adiabatically. As a result, (2.1.1) is contracted to a very simple universal equation which is sometimes called the Stuart-Landau equation. In fact, Landau was the first to conjecture the equation form (Landau, 1944), and Stuart was the first to derive it through an asymptotic method (Stuart, 1960). In quite a different context, specifically in laser theory, Haken and Sauermann (1963) derived a similar but more general equation. We shall outline in Sect. 2.2 how the Stuart-Landau equation is derived. The fact that dynamical systems can be reduced to some simple universal systems is by no means restricted to this particular bifurcation type. However, we do not intend in this book to present theories from such a general viewpoint. The method employed in Sect. 2.2 is a well-known multi-scale method, although there may be some possible variants leading to identical results. A practical use of the theory in Sect. 2.2 lies in the fact that it enables

us to calculate explicitly a certain constant (called the Landau constant) appearing in the Stuart-Landau equation, whose sign determines the stability of the bifurcating periodic solution. Otherwise, the Stuart-Landau equation itself is not likely to arouse much theoretical interest, although it may have some value in serving as an ideal nonlinear oscillator model.

So far, the discussion has been concerned with systems of *ordinary* differential equations. In many physical problems, *partial* differential equations describing processes in the space-time domain prove to be a more useful mathematical tool. For instance, one may mention the Navier-Stokes fluids, chemical reactions including diffusion, some ecological systems with migration, etc. Suppose that oscillatory motions occur in any of these continuous media as some control parameter is varied, and consider how to describe them. It is true that if the system is confined within a finite volume, the governing partial differential equations can, in principle, be transformed into a discrete set of ordinary differential equations, which describe the evolution of the amplitudes of the basis functions satisfying prescribed boundary conditions. Although the system then involves an infinite number of degrees of freedom, a mode-truncation approximation is usually allowed. Thus, as far as the onset of oscillations is concerned, there seems to be nothing theoretically new, compared to the bifurcation theory for systems of ordinary differential equations. Specifically, the application of a multi-scale method will lead to a Stuart-Landau equation again. (For a mathematical theory of the Hopf bifurcation for systems of partial differential equations in bounded domains, see Joseph and Sattinger, 1972; bifurcation analyses of reaction-diffusion systems have been developed by Auchmuty and Nicolis, 1975, 1976, and Herschkowitz-Kaufman, 1975.)

There may be some situations, however, where keeping to formal bifurcation theories easily makes us overlook a fact of considerable physical importance. The situation of particular interest in this connection seems to be when the system size is very large. Then, formal bifurcation techniques applied near μ_c cannot claim full validity except in an extremely limited parameter range about μ_c. This is basically because the eigenvalue spectrum obtained from the linearization about the reference steady state is almost continuous for large system size, so that, in addition to the couple of modes which are becoming unstable, a large number of degrees of freedom come into play as soon as μ deviates from μ_c (a more detailed description will be given in Sect. 2.3). Thus it is desirable that the Stuart-Landau equation be generalized so as to cover such circumstances. People in the field of fluid mechanics have developed theories in this direction, which proved to be very useful in understanding instabilities (not restricted to the Hopf type) arising in systems with large dimensions at least in one or two directions. Typical examples are the Newell-Whitehead theory (1969) on a fluid layer heated from below with infinite aspect ratio, and the Stewartson-Stuart theory (1971) on plane Poiseuille flow. In these theories, one works with partial differential equations throughout, not transforming them into ordinary differential equations. A method was contrived to reduce the equations to a generalized form of the Stuart-Landau equation, thereby admitting slow spatial and temporal modulation of the envelope of the bifurcating flow patterns. We call that equation the Ginzburg-Landau equation (named after a similar equation appearing in super-

conductivity) or the Stewartson-Stuart equation. In this book we adopt the former name.

Independently of the hydrodynamical context, the Ginzburg-Landau equation was derived by Graham and Haken (1968, 1970) in multimode lasers as a further development of the Haken-Sauermann theory (1963); it should be noted that fluctuations are included in most of their series of works. For various non-equilibrium phase transitions described by the Ginzburg-Landau-type equation, see the review article by Haken (1975b) and his more recent monograph (1983).

The derivation of the Ginzburg-Landau equation usually involves the method of multiple scales (in space and time), and again there are some variants in technical details. For convenience, we sometimes call all the related techniques involving the use of stretched space-time coordinates *the reductive perturbation method*, a term originally coined for a systematic method of deriving various nonlinear wave equations mainly in dissipationless media (Taniuti and Wei, 1968; Taniuti, 1974). It is now widely known that the Ginzburg-Landau equation is not only related to a few fluid mechanical or optical problems but that it can be deduced from a rather general class of partial differential equations (Newell, 1974; Haken, 1975a; Gibbon and McGuiness, 1981; Lin and Kahn, 1982). Chemical reactions with diffusion form a simple and particularly interesting class of systems in this connection (Kuramoto and Tsuzuki, 1974; Wunderlin and Haken, 1975), and we shall derive in Sect. 2.4 the Ginzburg-Landau equation for general reaction-diffusion systems. Just as the Stuart-Landau equation describes the simplest nonlinear oscillator, so the Ginzburg-Landau equation describes the simplest *field* of nonlinear oscillators. In later chapters, this equation will be frequently invoked in discussing chemical waves and chemical turbulence.

2.2 The Stuart-Landau Equation

In this section, we outline how a small-amplitude equation valid near the Hopf bifurcation point is derived from the general system of ordinary differential equations (2.1.1).

Let X and F be n-dimensional real vectors and μ a real scalar parameter. Let $X_0(\mu)$ denote a steady solution of (2.1.1) or

$$F(X_0(\mu);\mu) = 0 . \qquad \frac{d\vec{X}}{dt} = \vec{F}(\vec{X},\mu)$$

We express (2.1.1) in terms of the deviation $u \equiv X - X_0$ in a Taylor series:

$$\frac{du}{dt} = Lu + Muu + Nuuu + \dots , \tag{2.2.1}$$

where L denotes the Jacobian matrix whose ijth element is given by $L_{ij} = \partial F_i(X_0)/\partial X_{0j}$; the abbreviations Muu and $Nuuu$, etc., indicate vectors whose ith components are given by

$$(Muu)_i = \sum_{j,k} \frac{1}{2!} \frac{\partial^2 F_i(X_0)}{\partial X_{0j} \partial X_{0k}} u_j u_k ,$$

$$(Nuuu)_i = \sum_{j,k,l} \frac{1}{3!} \frac{\partial^3 F_i(X_0)}{\partial X_{0j} \partial X_{0k} \partial X_{0l}} u_j u_k u_l ,$$

and the higher-order terms in u may be expressed similarly. We shall later use quantities like Muv and $Nuvw$ for different vectors u, v and w, and their definitions may be understood as an obvious extension of the above. Note, in particular, that Muv and $Nuvw$ are symmetric functions of u, v and w. Note also that the expansion coefficients, which are symbolically expressed by M, N, etc., generally depend on μ at least through $X_0(\mu)$.

Suppose that μ is varied in some range about $\mu = 0$. We assume that up to $\mu = 0$ the solution X_0 remains stable to sufficiently small perturbations, while it loses stability for $\mu > 0$. Consider the linear eigenvalue problem associated with (2.2.1), i.e.,

$$Lu = \lambda u . \tag{2.2.2}$$

The stability of X_0 is related to the distribution of the eigenvalues λ in the complex plane. By assumption, this distribution changes with μ in the following way: all λ stay in the left half-plane if $\mu < 0$, and at least one eigenvalue crosses the imaginary axis at $\mu = 0$. Since the eigenvalues are given by the zeros of an nth-order polynomial with real coefficients, we have the following two general possibilities: (a) one eigenvalue on the real axis crosses the origin (Fig. 2.1a), (b) a pair of complex-conjugate eigenvalues cross the imaginary axis simultaneously (Fig. 2.1b). In each case, the eigenvalues are assumed to have nonzero transversal "velocity" when crossing the imaginary axis, or

$$\left. \frac{d \operatorname{Re}\{\lambda(\mu)\}}{d\mu} \right|_{\mu=0} > 0 .$$

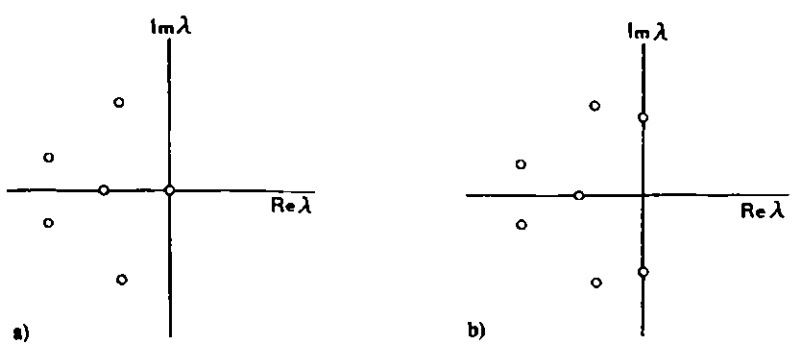

Fig. 2.1a, b. Two typical distributions of the eigenvalues at criticality

Furthermore, the rest of the eigenvalues are assumed to remain at a nonzero distance from the imaginary axis. In the following, we shall restrict our attention to case (b), since this corresponds to the Hopf bifurcation.

Near criticality, the matrix L may be developed in powers of μ:

$$L = L_0 + \mu L_1 + \mu^2 L_2 + \dots \,. \tag{2.2.3}$$

To save notation, let $\lambda(\mu)$ denote a special eigenvalue which is becoming critical rather than denoting a general one, and $\bar{\lambda}(\mu)$ its complex conjugate (we use a bar to signify a complex conjugate throughout). We assume a power-series expansion for λ also:

$$\lambda = \lambda_0 + \mu\lambda_1 + \mu^2\lambda_2 + \dots \,, \tag{2.2.4}$$

where λ_ν are generally complex, or $\lambda_\nu = \sigma_\nu + \mathrm{i}\,\omega_\nu$. By assumption,

$$\sigma_0 = 0\,, \qquad \sigma_1 > 0\,.$$

Let U denote the right eigenvector of L_0 corresponding to the eigenvalue $\lambda_0(-\mathrm{i}\,\omega_0)$:

$$L_0 U = \lambda_0 U\,, \qquad L_0 \bar{U} = \bar{\lambda}_0 \bar{U}\,.$$

Similarly, the left eigenvector is denoted by U^*:

$$U^* L_0 = \lambda_0 U^*\,, \qquad \bar{U}^* L_0 = \bar{\lambda}_0 \bar{U}^*\,,$$

where $U^*\bar{U} = \bar{U}^* U = 0$, and these vectors are normalized as $U^* U = \bar{U}^* \bar{U} = 1$. Note that λ_0 and λ_1 are expressed as

$$\lambda_0 = \mathrm{i}\,\omega_0 = U^* L_0 U\,, \tag{2.2.5a}$$

$$\lambda_1 = \sigma_1 + \mathrm{i}\,\omega_1 = U^* L_1 U\,. \tag{2.2.5b}$$

It is convenient to define a small positive parameter ε by $\varepsilon^2\chi = \mu$, where $\chi = \operatorname{sgn}\mu$; ε is considered to be a measure of the amplitude to lowest order, so that one may assume the expansion

$$u = \varepsilon u_1 + \varepsilon^2 u_2 + \dots \,. \tag{2.2.6}$$

profile 的改变小于本征矢量的改变

The expression in (2.2.3) now becomes

$$L = L_0 + \varepsilon^2\chi L_1 + \varepsilon^4 L_2 + \dots \,. \tag{2.2.7}$$

Similarly, for some higher-order expansion coefficients in (2.2.1), we write symbolically

$$M = M_0 + \varepsilon^2 \chi M_1 + \dots ,$$
$$N = N_0 + \varepsilon^2 \chi N_1 + \dots . \tag{2.2.8}$$

From the fact that λ has a small real part of order ε^2, it would be appropriate to introduce a scaled time τ via

$$\tau = \varepsilon^2 t ,$$

and regard u as depending both on t and τ, and having no explicit dependence on ε; t and τ will be treated as mutually independent. Correspondingly, the time differentiation in (2.2.1) should be transformed as

$$\frac{d}{dt} \to \frac{\partial}{\partial t} + \varepsilon^2 \frac{\partial}{\partial \tau} . \tag{2.2.9}$$

The substitution of (2.2.6 – 9) into (2.2.1) gives

$$\left(\frac{\partial}{\partial t} + \varepsilon^2 \frac{\partial}{\partial \tau} \quad L_0 - \varepsilon^2 \chi L_1 - \dots \right) (\varepsilon u_1 + \varepsilon^2 u_2 + \dots)$$
$$= \varepsilon^2 M_0 u_1 u_1 + \varepsilon^3 (2 M_0 u_1 u_2 + N_0 u_1 u_1 u_1) + O(\varepsilon^4) . \tag{2.2.10}$$

Equating coefficients of different powers of ε in (2.2.10), we have a set of equations in the form

$$\left(\frac{\partial}{\partial t} - L_0 \right) u_\nu = B_\nu , \qquad \nu = 1, 2, \dots . \tag{2.2.11}$$

The first few B_ν are

$$B_1 = 0 , \tag{2.2.12a}$$

$$B_2 = M_0 u_1 u_1 , \tag{2.2.12b}$$

$$B_3 = - \left(\frac{\partial}{\partial \tau} - \chi L_1 \right) u_1 + 2 M_0 u_1 u_2 + N_0 u_1 u_1 u_1 . \tag{2.2.12c}$$

In general, the B_ν are functions of the lower-order quantities $u_{\nu'}$ $(\nu' < \nu)$.

For the system of linear inhomogeneous equations (2.2.11), we have an important property:

$$\int_0^{2\pi/\omega_0} U^* \cdot B_\nu e^{-i\omega_0 t} dt = 0 , \tag{2.2.13}$$

which follows from

$$\int_0^{2\pi/\omega_0} U^* \cdot B_\nu e^{\,i\omega_0 t} dt = \int_0^{2\pi/\omega_0} \left[U^* \cdot \left(\frac{\partial}{\partial t} - L_0 \right) u_\nu \right] e^{-i\omega_0 t} dt$$
$$= \int_0^{2\pi/\omega_0} (i\omega_0 U^* \cdot u_\nu - i\omega_0 U^* \cdot u_\nu) e^{-i\omega_0 t} dt = 0 .$$

The equality (2.2.13) is called the solvability condition. By inspecting the general structure of (2.2.11), it is expected that the u_ν can be found iteratively as 2π-periodic functions of $\omega_0 t$. This means that $B_\nu(t, \tau)$ is also 2π-periodic in $\omega_0 t$, so that it would be appropriate to express it in the form

$$B_\nu(t, \tau) = \sum_{l=-\infty}^{\infty} B_\nu^{(l)}(\tau) e^{il\omega_0 t} .$$

The solvability condition now reduces to

$$U^* \cdot B_\nu^{(1)}(\tau) = 0 , \tag{2.2.14}$$

which is the crucial condition used below.

For $\nu = 1$, (2.2.11) may be solved in the form

$$u_1(t, \tau) = W(\tau) U e^{i\omega_0 t} + \text{c.c.} , \tag{2.2.15}$$

where c.c. stands for the complex conjugate, and $W(\tau)$ is some complex amplitude yet to be specified. This is called the neutral solution. We shall soon find later that the evolution equation for W, which is nothing but the Stuart-Landau equation, is given by (2.2.14) for $\nu = 3$, or by

$$U^* \cdot B_3^{(1)} = 0 . \tag{2.2.16}$$

Note that (2.2.14) is trivially satisfied for $\nu = 2$ because $B_2^{(1)}$ vanishes identically as is clear from (2.2.12b) and (2.2.15). In order to derive the equation obeyed by W, it is thus necessary to express u_2 appearing in $B_3^{(1)}$ in terms of u_1 (or W), and this can be done by solving (2.2.11) for u_2 as a function of u_1. But B_2 contains only the zeroth and second harmonics, and the same is true for u_2. Thus, we try to find u_2 in the form

$$u_2 = V_+ W^2 e^{2i\omega_0 t} + V_- \bar{W}^2 e^{-2i\omega_0 t} + V_0 |W|^2 + v_0 u_1 . \tag{2.2.17a}$$

By substituting this into (2.2.11) for $\nu = 2$, the quantities $V_{\pm,0}$ are obtained in the form

$$V_+ = \bar{V}_- = -(L_0 - 2i\omega_0)^{-1} M_0 U U ,$$
$$V_0 = -2 L_0^{-1} M_0 U \bar{U} . \tag{2.2.17b}$$

The constant v_0 cannot be determined at this stage,but we do not need it for the present purpose. Substituting (2.2.15, 17a) into (2.2.12c), we have

$$B_3^{(1)} = -\left(\frac{\partial}{\partial\tau} - \chi L_1\right) WU + (2M_0 U V_0 + 2M_0 \bar{U} V_+ + 3N_0 U U \bar{U}) |W|^2 W. \tag{2.2.18}$$

Then the solvability condition (2.2.16) itself turns out to take the form of the Stuart-Landau equation

$$\frac{\partial W}{\partial\tau} = \chi\lambda_1 W - g|W|^2 W, \tag{2.2.19}$$

where g is a complex number given by

$$g \equiv g' + \mathrm{i}g'' = -2U^* M_0 U V_0 - 2U^* M_0 \bar{U} V_+ - 3U^* N_0 U U \bar{U}. \tag{2.2.20}$$

Defining the amplitude R and the phase Θ via $W = R\exp(\mathrm{i}\Theta)$, one may alternatively write (2.2.19) as

$$\begin{aligned} \frac{dR}{d\tau} &= \chi\sigma_1 R - g' R^3, \\ \frac{d\Theta}{d\tau} &= \chi\omega_1 \quad g'' R^2. \end{aligned} \tag{2.2.21}$$

The non-trivial solution

$$R = R_s, \qquad \Theta = \tilde{\omega} t + \text{const},$$

$$R_s = \sqrt{\sigma_1/|g'|}, \qquad \tilde{\omega} = \chi(\omega_1 - g'' R_s^2),$$

appears only in the supercritical region ($\chi > 0$) for positive g' and in the subcritical region for negative g'. In the former case, the bifurcation is called supercritical, and in the latter case, subcritical. Linearization about R_s shows that the supercritical bifurcating solution is stable, while the subcritical one is unstable. The bifurcating solution shows a perfectly smooth circular motion in the complex W plane. The corresponding expression for the original vector variable X is approximately given by

$$X \simeq X_0 + \varepsilon u_1 = X_0 + \varepsilon\{U R_s \exp[\mathrm{i}(\omega_0 + \varepsilon^2\tilde{\omega})t] + \text{c.c.}\},$$

which describes a small-amplitude elliptic orbital motion in the critical eigenplane.

2.3 Onset of Oscillations in Distributed Systems

The foregoing argument was about systems of ordinary differential equations. For chemical reactions this corresponds to the dynamics of local systems which